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Nonlinear behavior of the renormalization group flow and standard model parameters

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Abstract

The standard model for high-energy physics (SM) describes fundamental interactions between subatomic particles down to a distance scale on the order of 10^{-18} m. Despite its widespread acceptance, a consistent and comprehensive understanding of SM parameters is missing. Starting from a less conventional standpoint, our work suggests that the spectrum of particle masses, gauge couplings and fermion mixing angles may be derived from the chaotic regime of the renormalization group flow. In particular, we argue that the observed hierarchies of standard model parameters amount to a series of scaling ratios depending on the Feigenbaum constant. Leading order predictions are shown to agree well with experimental data.

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1. Introduction

The generation structure of quarks and leptons stands out as one of the most intriguing puzzles of the standard model for particle physics (SM). The conventional formulation of the SM requires 19 free input parameters, among which 12 can be expressed in terms of empirical mass eigenvalues [1]. In addition, there is a set of four inputs determined by the so-called Cabibbo–Kobayashi–Maskawa (CKM) matrix whose structure includes three quark-mixing angles and one CP phase [18,19]. The remaining three parameters are two gauge couplings (α_3, α_{em}) and the strong CP phase. Recent experiments in neutrino physics have confirmed the existence of neutrino oscillations and masses and have subsequently triggered a host of challenging questions [2–4]. There is a large body of proposed extensions of SM, each of them attempting to resolve some unsatisfactory aspects of the theory while introducing new unknowns. In contrast with the line of thought pursued by

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these models, our work suggests that the spectrum of particle masses, gauge couplings and fermion mixing angles may be derived from the *chaotic behavior of the renormalization group* (*RG*) flow. Although predictions are found to match reasonably well experimental observations, we caution that our results are entirely preliminary and a concurrent analysis is needed to confirm or disprove their validity.

The standard procedure for investigating the high-energy domain of any effective field theory is to start from the underlying RG flow equations, identify its fixed points and analyze the asymptotic flow of coupling parameters in the basin of attraction of these points [5]. Taking an alternative approach, we treat the RG flow equation as a generic iterated mapping and evaluate its chaotic regime after a large number of iteration cycles. We conclude that the observed hierarchy of SM parameters amounts to a series of scaling ratios depending on the Feigenbaum constant [27]. Since fermion mass scaling ratios and mixing matrices can be parameterized in terms of the Cabibbo angle [6–8,28], this result supplies a natural connection between the Cabibbo angle and the Feigenbaum constant. Moreover, it is found that the model can accommodate hypothetical generations of both heavy and ultra-light fermions that are expected to emerge beyond the energy range of SM. A representative example in this regard is the fourth SM family neutrino whose detection is anticipated at future linear colliders [9].

The paper is organized in the following way: Section 2 outlines the background of the RG flow equation and derives the asymptotic link between the beta-function and Feigenbaum scaling for a generic effective field theory. The emergence of a hierarchical pattern of observables based on this link is elaborated upon in Section 3 with specifics on SM hierarchies detailed in Section 4. The last three sections include a brief presentation of future extensions, open questions and concluding remarks.

2. Beta-function and Feigenbaum scaling

Following the framework of RG transformations, all physical observables of an effective field theory can be formulated in terms of a finite number of renormalized couplings [10]. These are defined at an arbitrary mass scale μ referred to as a "subtraction point" or "sliding scale". One key result of RG is that any change in the renormalized correlation functions in response to a variation in μ must be compensated by a corresponding change in the renormalized couplings. The outcome of this conjecture is contained in the so-called Callan–Symanzik equation, which reflects how all observables of the theory change (or "flow") with μ. Beta-function of the renormalization group flow is defined by the partial differential equation

$$\beta[g(\mu)] \doteq \mu \frac{\partial g(\mu)}{\partial \mu} \tag{1}$$

The zeroes of the beta-function, generically called fixed points, are of particular interest in the theory of RG flows. Knowledge of the fixed points enables the study of high and low-energy domains of the effective field theory [10,11].

We proceed from these preliminary considerations by introducing the following set of working assumptions:

1. The effective field theory contains a single coupling parameter $g = g(\mu)$.
2. The asymptotic flow of the coupling parameter toward the fixed point g^* reflects the approach to the high-energy domain of field theory.
3. The phase transition associated with the flow $g(\mu) \to g^*$ is an infinite-order phase transition.

The last two assumptions may be linked to framework of conformal field theories, which are considered well suited for the description of high-energy physics [12–14]. A remarkable feature of infinite-order phase transitions is that the correlation length ξ has an essential singularity at the critical coupling g^* given by

$$\xi \sim \exp(A|g - g^*|^{-\sigma}) \tag{2}$$

in which σ is a critical exponent and A a constant. Such behavior develops when the coupling parameter has a vanishing mass dimension at g^* and the beta-function may be represented as a quadratic function of g [12–14]

$$\mu\frac{\partial g}{\partial\mu} = cg^2 + O(g^3) \tag{3}$$

where c is a real-valued coefficient. The discrete analogue of (3) reads

$$g_{n+1} - g_n = \frac{c\Delta\mu}{\mu}g_n^2 + O(g_n^3) \tag{4}$$

Here n is the iteration index and the subtraction point increment $\Delta\mu$ represents a scalar fixed by resolution requirements. Any realistic description of the RG flow in the high-energy domain must take into account statistical fluctuations stemming from the uncertainty principle. Because large fluctuations and non-equilibrium microscopic processes dominate the physics on short time scales, the temporal resolution $\Delta t_n \doteq t_{n+1} - t_n$ is expected to vary as the inverse of time measurement, i.e. $\Delta t_n \sim t_n^{-1}$. It follows that the dimensionless subtraction point entering (4) and defined as $\tilde{\mu} \doteq \mu/c\Delta\mu$ acts as an autonomous control parameter. Following the onset of chaos in quadratic maps through period doubling bifurcations, it can be shown that the transition from a period 2^n super stable orbit to a period 2^{n+1} super stable orbit occurs for a geometrically spaced series of control parameters given by [15]

$$\tilde{\mu}_n - \tilde{\mu}_\infty \sim \delta_2^{-n} \tag{5}$$

where $2^n \gg 1$ and where $\delta_2 \doteq 4.669\ldots$ is the Feigenbaum constant for the quadratic map. From the previous discussion it can be inferred that $\tilde{\mu}_\infty$ represents the fixed point of the $\tilde{\mu}_n$ series whose generic term is defined via

$$\tilde{\mu}_n \doteq \frac{\mu}{c\Delta\mu_n} \sim \frac{\mu}{c}t_n^{-1} \tag{6}$$

It is important to emphasize that (5) is frame-independent, in the sense that its form is not affected by changing the subtraction point and its limit $\tilde{\mu}'_n = s\tilde{\mu}_n$, $\tilde{\mu}'_\infty = s\tilde{\mu}_\infty$ with $s \in \{R\}$.

To streamline the derivation and without losing generality, we further assume that a plausible boundary condition in (5) is $\tilde{\mu}_\infty \approx 0$. This ansatz may be justified by considering that the RG flow develops over sufficiently large times ($t_n \gg 0$).

The emergence of scaling (5) points out to an important result regarding the asymptotic form of the beta-function. According to the guiding prescription of RG analysis, the evolution of the beta-function may be studied through a sequence of renormalization steps consisting of iterated composition and rescaling operations [15,16]. Let $\tilde{\beta}(g)$ designate the universal Feigenbaum–Cvitanovic function that satisfies the so-called renormalization equation

$$\tilde{\beta}(g) \doteq -\alpha\tilde{\beta}\left(\tilde{\beta}\left(\frac{g}{\alpha}\right)\right) \tag{7}$$

in which $\alpha \doteq 2.5029\ldots$ After a large number of iteration cycles ($2^n \gg 1$), the renormalized beta-function $\tilde{\beta}_n(g) \doteq (-\alpha)^n\beta^{(2^n)}(\frac{g}{\alpha^n}, \tilde{\mu}_n)$ approaches $\tilde{\beta}(g)$ according to [15,16]

$$\tilde{\beta}(g) = \lim_{n\to\infty}(-\alpha)^n\beta^{(2^n)}\left(\frac{g}{\alpha^n}, \tilde{\mu}_\infty\right) \tag{8}$$

The renormalized beta-function obeys the recursive relation

$$\tilde{\beta}_{n-1}(g) \doteq -\alpha\tilde{\beta}_n\left(\tilde{\beta}_n\left(\frac{g}{\alpha}\right)\right) \tag{9}$$

such that

$$\tilde{\beta}_n(g) - \tilde{\beta}(g) \sim \delta_2^{-n}h(g) \tag{10}$$

where $h(g)$ is an analytic function. Moreover, since our focus is the coupling flow in the immediate neighborhood of g^*, where $(g - g^*) \sim O(\varepsilon)$, we may reasonably assume that $\tilde{\beta}(g) \sim O(\varepsilon)$. We arrive at

$$\tilde{\beta}_n(g) \sim \delta_2^{-n}h(g) \tag{11}$$

The above power-law behavior reveals the asymptotic connection between the renormalized beta-function, on the one hand, and Feigenbaum constant on the other. Next sections explore the impact of this result on key observables describing a typical field theoretic framework such as the electroweak model or QCD.

3. Hierarchical pattern of observables

Let Ω be a generic observable of the effective field theory such as mass, gauge coupling or mixing angle. Assuming that the dimension of Ω is d_0, that is

$$[\Omega] = [\mu]^{d_0} \tag{12}$$

we may write, by dimensional analysis [17]

$$\Omega(\mu, g(\mu)) = \mu^{d_0} f_{d_0}(g(\mu)) \tag{13}$$

Constraining the function $f_{d_0}(g(\mu))$ to be independent of the subtraction point yields

$$\frac{d\Omega(\mu, g(\mu))}{d\mu} = 0 \rightarrow f_{d_0}(g(\mu)) \sim \exp\left(-d_0 \int_{g^*}^{g(\mu)} \frac{dg}{\tilde{\beta}(g)}\right) \tag{14}$$

On account of the RG interpretation previously developed, the dimensionless form of (13) may be written as

$$\Omega_n(\tilde{\mu}_n, g(\tilde{\mu}_n)) \sim \tilde{\mu}_n^{d_0} \exp[-\delta_2^n d_0 F(g(\tilde{\mu}_n), g^*)] \tag{15}$$

with

$$F(g(\tilde{\mu}_n), g^*) \doteq \int_{g^*}^{g(\tilde{\mu}_n)} \frac{dg}{h(g)} \tag{16}$$

The integral (16) may be approximated around $\tilde{\mu}_\infty \approx 0$ as

$$F(g(\tilde{\mu}_n), g^*) \approx \frac{g(\tilde{\mu}_n) - g^*}{h(g^*)} \approx \tilde{\mu}_n \frac{\left[\frac{\partial g}{\partial \tilde{\mu}_n}(0)\right]}{h(g^*)} = \delta_2^{-n} \frac{\left[\frac{\partial g}{\partial \tilde{\mu}_n}(0)\right]}{h(g^*)} \tag{17}$$

which implies that, for two arbitrary iteration indices,

$$\frac{\Omega_n(\tilde{\mu}_n, g(\tilde{\mu}_n))}{\Omega_m(\tilde{\mu}_m, g(\tilde{\mu}_m))} \sim \delta_2^{(m-n)d_0} \tag{18}$$

We end this section by noting that $d_0 = 1$ if the observable (13) refers to a mass parameter and $d_0 \sim O(\varepsilon)$ if it refers to a gauge coupling or a mixing angle. The latter property is a direct consequence of (2) which implies that coupling charges behave as marginal parameters in the immediate neighborhood of g^* [12,13]. In this case it is reasonable to assume that, on a first-order basis, the index difference $(m - n)d_0$ for $m,n \gg 1$ may be rounded off to the closest integer.

4. Scaling hierarchies of standard model parameters

A remarkable yet unexplained property of SM parameters is that they appear to be organized in a hierarchical fashion. The scaling ratio of two parameters in the hierarchy depends on integer powers of the Cabibbo angle whose experimental best-fit value is $\theta_C = 12.9$–$13°$ [23]. It is customary to work with the Cabibbo angle in the equivalent trigonometric form, that is, $\lambda \doteq \sin\theta_C = 0.223$–$0.225$. Let us assume that the set of charged lepton and current-quark masses, evaluated at an arbitrary energy scale, are denoted by the vector M_l and matrix M_q, respectively

$$M_l \doteq [m_e \quad m_\mu \quad m_\tau] \quad M_q \doteq \begin{bmatrix} m_u & m_d \\ m_c & m_s \\ m_t & m_b \end{bmatrix} \tag{19}$$

The explicit set of scaling ratios in (19) is given by [6–8,21,26]:

$$\begin{aligned} &\frac{m_e}{m_\mu} \sim \lambda^4 \quad \frac{m_\mu}{m_\tau} \sim \lambda^2 \\ &\frac{m_c}{m_t} \sim \lambda^4 \quad \frac{m_s}{m_b} \sim \lambda^2 \\ &\frac{m_u}{m_t} \sim \lambda^8 \quad \frac{m_d}{m_b} \sim \lambda^4 \end{aligned} \tag{20}$$

We note that the pattern of charged fermion masses depends on integer powers of λ^2. Here, quark masses are arranged in two columns each involving three independent flavors, namely (u,c,t) and (d,s,b).

It is pertinent to bring up at this point the issue of fermion mixing and its parameterization. As it is known, in the SM quark mass eigenstates are different from their weak eigenstates partners and the CKM matrix, denoted by V_{CKM}, relates these two bases by operating on the $(-1/3)$ mass states (d,s,b) [18,19]:

$$\begin{bmatrix} d' \\ s' \\ b' \end{bmatrix} = V_{\mathrm{CKM}} \begin{bmatrix} d \\ s \\ b \end{bmatrix} \tag{21}$$

In terms of individual mixing components, we have

$$V_{\mathrm{CKM}} \doteq \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} \tag{22}$$

Unlike (20), the CKM matrix expressed using the so-called Wolfenstein parameterization [20] is approximated to the leading order by entries dependent on integer powers of λ[1]

$$V_{\mathrm{CKM}} \approx \begin{vmatrix} 1 & \lambda & \lambda^3 \\ -\lambda & 1 & \lambda^2 \\ \lambda^3 & -\lambda^2 & 1 \end{vmatrix} \tag{23}$$

A similar matrix structure may be assigned to the recently discussed set of operators describing mixing in the lepton sector [21,22]. Specifically, if the neutrino mass matrix m_ν and the charged lepton mass matrix m_l are diagonalized through the following transformations

$$\begin{aligned} m_\nu &\doteq U_\nu m_\nu^{\mathrm{diag}} U_\nu^T \\ m_l &\doteq U_L m_l^{\mathrm{diag}} U_R^+ \end{aligned} \tag{24}$$

then it can be shown that neutrino mixing, defined by the so-called Pontercovo–Maki–Nakagawa–Sakata (PMNS) matrix, may be represented as

$$U_{\mathrm{PMNS}} = U_L^+ U_\nu \tag{25}$$

In one plausible scenario, one finds [21]

$$\begin{aligned} m_l m_l^+ &\sim m_\tau^2 \begin{vmatrix} \lambda^6 & \lambda^5 & \lambda^3 \\ \lambda^5 & \lambda^4 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{vmatrix} \\ U_L m_l^{\mathrm{diag}} U_L^T &\sim m_\tau \begin{vmatrix} \lambda^4 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{vmatrix} \end{aligned} \tag{26}$$

The standard parameterization of the PMNS matrix is formulated with the help of three mixing angles $(\theta_{12}, \theta_{23}, \theta_{13})$. According to the above scenario, we have

$$\begin{aligned} \sin\theta_{12} &= \lambda \\ \sin\theta_{13} &= A\lambda^3 \\ \sin\theta_{23} &= B\lambda^2 \end{aligned} \tag{27}$$

where A, B are positive numbers of order unity.

[1] To simplify the argument, CP-violating phases are neglected here.

Finally, it is instructive to recall that SM coupling charges and weak boson masses satisfy the following scaling pattern [23]

$$\left(\frac{e}{g_2}\right)^2 \sim \lambda_W \quad \left(\frac{e}{g_{3s}}\right)^2 \sim \lambda_W^2$$
$$\left(\frac{M_W}{M_Z}\right)^2 \sim 1 - \lambda_W \tag{28}$$

Here, $e^2 \doteq 4\pi\alpha_{em}$, $g_2^2 \doteq 4\pi\alpha_2$, $g_{3s}^2 \doteq 4\pi\alpha_{3s}$ stand for the electromagnetic, weak and strong coupling charges and λ_W is the "sine" squared of the Weinberg angle, whose magnitude is nearly identical to the nominal value of the Cabibbo angle ($\lambda_W \doteq \sin^2\theta_W = 0.229$) [10,23].

As stated at the beginning of this section, relationships (19)–(28) provide ample analytical evidence that SM parameters display a hierarchical dependence on the Cabibbo angle. This observation is consistent with (18) and strongly suggests a direct connection between λ and δ_2. In fact

$$\lambda \sim \delta_2^{-1} = 0.214\ldots \tag{29}$$

which leads one to conclude that the Feigenbaum constant for the quadratic map plays a central role in the observed patterns of particle masses, gauge couplings and fermion mixing angles.[2]

5. Future extensions

As it is known, the SM does not fix the number of fermion families. For example, current data allow for additional generations of leptons and quarks if the mass of the fourth family neutrino is larger than $M_Z/2$ [9]. At the other end of the energy scale, various studies on neutrinoless double beta-decay processes point to a spectrum of ultra-light neutrinos with masses well below the eV threshold [25]. As the ladder-like pattern of SM parameters encoded in (18) and (19) is not bounded by fixed limits on the index difference $(n-m)d_0$, one may infer that new fermion generations arise beyond what is known today. The object of this section is to formulate first-order predictions on the hypothetical ultra-light and super-heavy fermion masses that may be observed in future experiments. The most straightforward extrapolation of (20) on account of (29) gives

$$\begin{aligned} m_{l4} &\sim m_{\nu_e}\delta_2^{-2} < 4.6 \times 10^{-2}\ \text{eV} \\ m_{l5} &\sim m_\tau \delta_2^2 = 38.76\ \text{GeV} \\ m_{q4} &\sim m_u \delta_2^{-2} = 0.107\ \text{eV} \\ m_{q5} &\sim m_t \delta_2^2 = 3.95\ \text{TeV} \end{aligned} \tag{30}$$

Here, $l4$, $q4$ ($l5$, $q5$) denote the ultra-light (super-heavy) families of leptons and quarks, respectively, whereas $m_{\nu_e}(< 1\ \text{eV})$, m_τ, m_u, m_t are best-fit fermion masses evaluated at the Z boson scale [9,26]. We find that these numbers agree well with predictions derived from the models developed in [9,25].

6. Open questions

The primary goal of this work was to present arguments that support an unexpected connection between the Feigenbaum scaling and RG, on the one hand, and SM hierarchies, on the other. Needles to say, our study does not provide a rigorous and comprehensive account of the physics underpinning the generation structure of the SM. Many questions remain open. Their satisfactory resolution requires a more extensive and refined plan of attack as well as a wealth of currently unavailable experimental data. Although a complete list of questions is not a practical option, we believe that among the most pressing issues that need to be dealt with are the following:

[2] A similar scenario is analyzed in [24] where mass generation in the lepton sector arises from the dissipative chaotic dynamics of the basic weak boson-fermion system.

1. What explains the small numerical difference $\lambda - \delta_2^{-1} \approx 9 \times 10^{-3}$? Are contributions related to higher non-linear terms in (3) and (4) relevant to this context?
2. Why is the mass hierarchy dependent on integer powers of δ_2^{-2} whereas the gauge coupling and mixing angle hierarchy depend on integer powers of δ_2^{-1}?
3. What mechanism is responsible for maintaining the parameter hierarchy in the transition from the high-energy domain of field theory to the low-energy domain of the SM?

7. Concluding remarks

We have suggested that the chaotic behavior of the RG flow offers valuable insights into the generation puzzle of the SM. In particular, it was argued that the observed hierarchies of standard model parameters amount to a series of scaling ratios depending on the Feigenbaum constant. A direct link was found between this constant and the Cabibbo angle. Future generations of "would-be" heavy and ultra-light fermions may be extrapolated using this dynamical model.

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